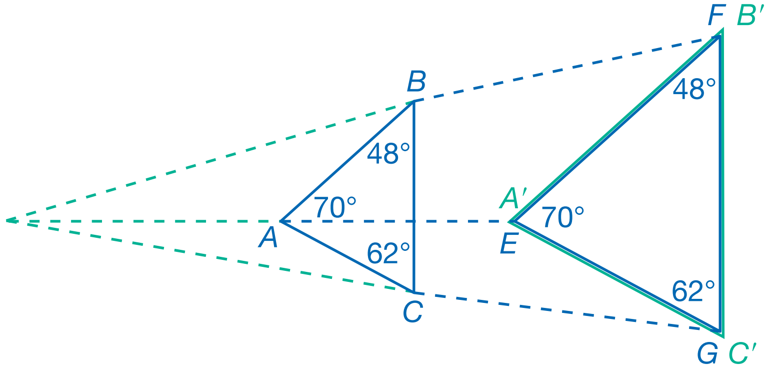
LESSON 11

Similar Triangles

Similarity with Angles

UNDERSTAND Two triangles are similar if all of their corresponding angles have equal measures and all of their corresponding sides have proportional lengths. However, you do not need to know every one of those angle measures and side lengths to prove that two triangles are similar.

Each angle in $\triangle ABC$ is congruent to an angle in $\triangle EFG$. However, none of the lengths of the sides of either triangle are known. If there is a dilation that carries $\triangle ABC$ on to $\triangle EFG$, then the triangles must be similar. The dilated image $\triangle A'B'C'$ exactly covers $\triangle EFG$, so $\triangle ABC$ must be similar to $\triangle EFG$.

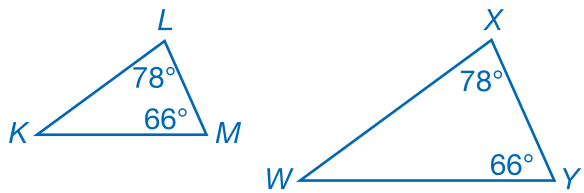


Dilating a figure produces a similar figure with congruent corresponding angles. The reverse is also true. If all corresponding angles in two triangles are congruent, dilating one triangle can produce the other triangle, so the triangles must be similar.

Notice that for figures with four or more sides, two figures can have identical angles but still not be similar. Consider a square and a rectangle. Even though the figures have the same set of angles, all right angles, they are not similar.

UNDERSTAND The sum of the measures of the interior angles of any triangle is 180°. If you know the measure of any two angles in a triangle, the measure of the third angle can have only one value.

Triangles *KLM* and *WXY* both contain a 78° angle and a 66° angle. In order for the sums of the angles to be 180°, the third angle of both triangles must measure 36°. The triangles have identical sets of angles, so they are similar triangles.

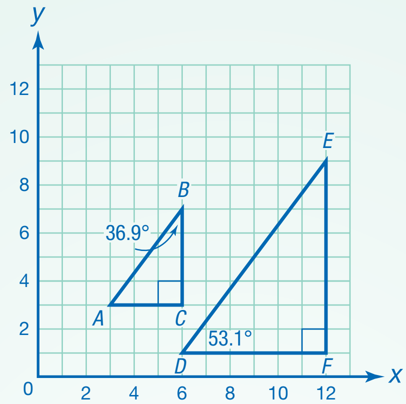


As you can see, knowing two angles in each triangle is enough information to determine if the triangles are similar. This fact is the basis for the postulate defined below.

Angle-Angle Similarity (AA~) Postulate: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

Connect

Prove that $\triangle ABC \sim \triangle DEF$ by the $AA\sim$ Postulate.



1 Identify corresponding angles.

If $\triangle ABC \sim \triangle DEF$, the order of the letters in each triangle's name will identify corresponding angles. So, $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$, and $\angle C$ corresponds to $\angle F$.

2 Show that one pair of corresponding angles are congruent.

$\angle C \cong \angle F$ because both measure 90°.

3 Show that a second pair of corresponding angles are congruent.

To determine if $\angle A \cong \angle D$, find $m\angle A$.

$$m\angle A + m\angle B + m\angle C = 180°$$
$$m\angle A + 36.9° + 90° = 180°$$
$$m\angle A = 180° - 36.9° - 90°$$
$$m\angle A = 53.1°$$

This is equal to the measure of $\angle D$, so the triangles have two pairs of congruent corresponding angles.

▶ Since $\angle A \cong \angle D$ and $\angle C \cong \angle F$, then $\triangle ABC \sim \triangle DEF$ by the $AA\sim$ Postulate.

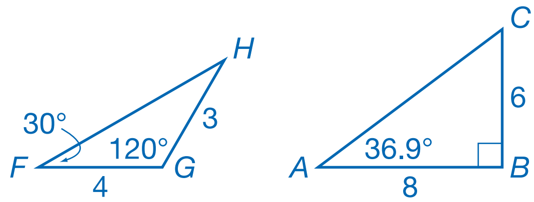
CHECK

Check that the triangles are similar by finding a center of dilation and a scale factor that would transform $\triangle ABC$ into $\triangle DEF$.

Similarity with Sides and Angles

UNDERSTAND Suppose that instead of knowing the measures of the angles in a pair of triangles, you know the length of their sides. Is this enough information to determine if the triangles are similar?

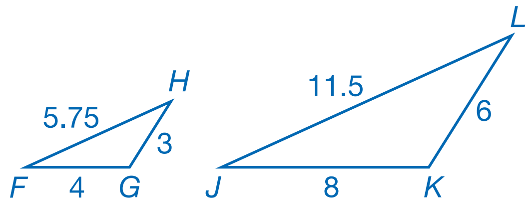
Unlike with angle measures, showing that two pairs of corresponding sides are proportional in length does not prove that two triangles are similar. For example, in $\triangle FGH$ and $\triangle ABC$, two pairs of sides are in the same ratio.



$$\frac{AB}{FG} = \frac{8}{4} = 2 \qquad \frac{BC}{GH} = \frac{6}{3} = 2$$

However, they are not the same shape, and they do not have corresponding angles that are congruent. They are not similar.

Now, consider $\triangle FGH$ and $\triangle JKL$. In these triangles, all three pairs of sides are in the same ratio.



$$\frac{JK}{FG} = \frac{8}{4} = 2 \qquad \frac{KL}{GH} = \frac{6}{3} = 2 \qquad \frac{JL}{FH} = \frac{11.5}{5.75} = 2$$

Think of the three sides of $\triangle FGH$ as three line segments. Dilating $\overline{FG}$, which has a length of 4 units, by a scale factor of 2 will produce a line segment with a length of 8 units. Dilating $\overline{GH}$, with length 3, by the same scale factor will produce a line segment with a length of 6 units. Dilating $\overline{FH}$, with length 5.75, by this same scale factor will produce a line segment with a length of 11.5 units. If these dilated line segments are assembled to form a triangle, that triangle will be identical to $\triangle JKL$.

As you can see, knowing the lengths of all of the sides of two triangles allows you to determine if they are similar. This is summarized in a mathematical theorem.

Side-Side-Side Similarity (SSS~) Theorem: If the three sides of one triangle are proportional in length to the three sides of another triangle, then the triangles are similar.

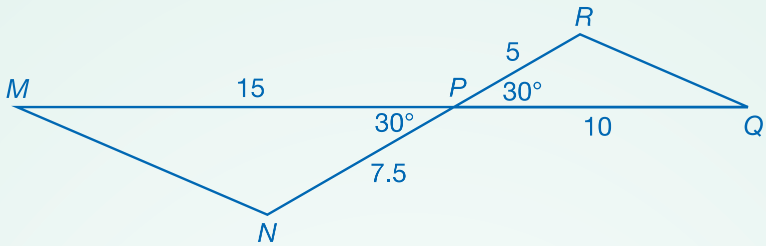
UNDERSTAND As you saw above, showing that only two pairs of corresponding sides are proportional in length is not enough information to prove that two triangles are similar. However, if two pairs of sides are proportional in length and the angles between those sides have equal measures, then the two triangles must be similar.

Side-Angle-Side Similarity (SAS~) Theorem: If two sides of one triangle have lengths that are proportional to two sides of another triangle and the included angles of those sides are congruent, then the triangles are similar.

On the next page, we will verify the SAS~ Theorem.

Connect

Use what you know about dilations to show that $\triangle MNP \sim \triangle QRP$, according to the SAS~ Theorem.



1

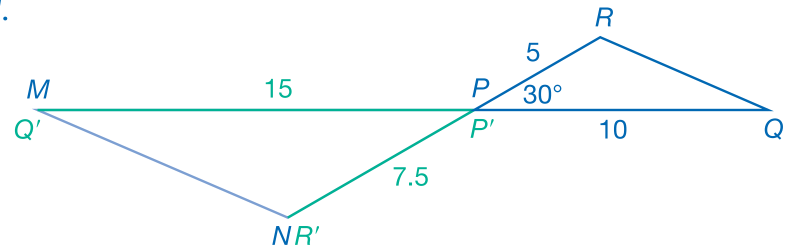
Dilate two sides of $\triangle QRP$.

Treat the sides as separate line segments. Dilate $\overline{PR}$ by a scale factor of -1.5 from vertex P. For a negative scale factor, draw the image on the opposite side of the center of dilation. Notice that the dilated segment $\overline{P'R'}$ covers $\overline{PN}$.

$5 \cdot (-1.5) = -7.5$

Dilate $\overline{PQ}$ by a scale factor of -1.5 from point P. Notice that the dilated segment $\overline{P'Q'}$ covers $\overline{PM}$.

$10 \cdot (-1.5) = -15$

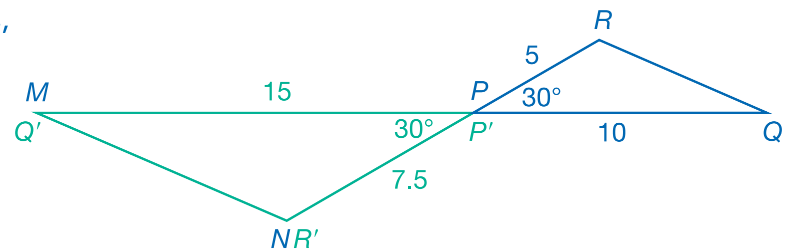


2

Complete the triangle.

Notice that, since they are dilations, $\overline{P'Q'}$ and $\overline{P'R'}$ form the same angle as $\overline{PQ}$ and $\overline{PR}$.

There is only one possible line segment that has endpoints Q' and R'. That segment, $\overline{Q'R'}$, covers $\overline{MN}$.

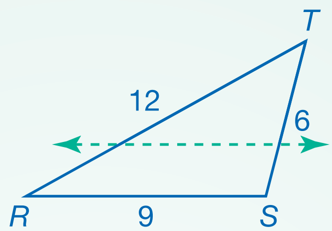


▶ $\overline{PR} \sim \overline{PN}$, $\overline{PQ} \sim \overline{PM}$, and $\angle QPR \cong \angle MPN$, so $\triangle MNP \sim \triangle QRP$.

DISCUSS

If $\triangle QRP$ was dilated by positive 1.5, would the image be identical to $\triangle MNP$?

EXAMPLE A Use $\triangle RST$ to show that a line parallel to one side of a triangle divides the other two sides proportionally.



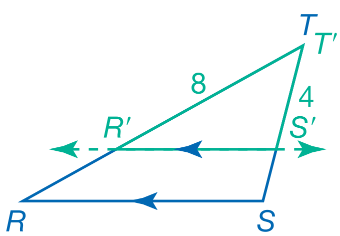
1 Dilate $\triangle RST$ from point T.

Begin by dilating $\overline{TR}$ by a factor of $\frac{2}{3}$. This produces segment $\overline{T'R'}$, which has length $\frac{2}{3} \cdot 12 = 8$ units.

Then dilate $\overline{TS}$ by the same scale factor. This produces segment $\overline{T'S'}$, which has length $\frac{2}{3} \cdot 6 = 4$ units.

Complete the image by drawing in segment $\overline{R'S'}$ to form dilated triangle $R'S'T'$.

When a polygon is dilated, corresponding sides are always parallel (or collinear). So, $\overline{R'S'}$ must be parallel to $\overline{RS}$.



Notice that $\triangle RST \sim \triangle R'S'T'$ according to the Side-Angle-Side Similarity Theorem.

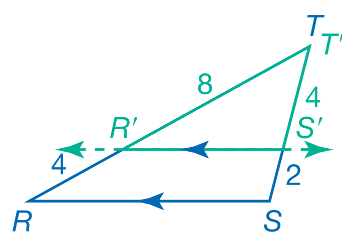
2 Determine if the sides are divided proportionally.

Find the lengths of $R'R$ and $S'S$.

$TR' + R'R = TR$ $\quad$ $TS' + S'S = TS$

$8 + R'R = 12$ $\quad$ $4 + S'S = 6$

$R'R = 4$ $\quad$ $S'S = 2$



Compare line segments to see if the sides are partitioned in the same ratio.

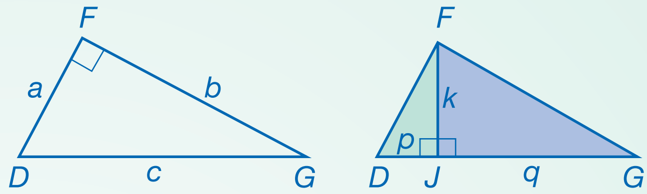
$\frac{TR'}{R'R} = \frac{8}{4} = 2 \qquad \frac{TS'}{S'S} = \frac{4}{2} = 2$

▶ A line parallel to $\overline{RS}$ divides the other two sides proportionally.

DISCUSS

Suppose you had dilated $\triangle RST$ by a different scale factor. Would the sides intersected by the line parallel to $\overline{RS}$ still be divided proportionally?

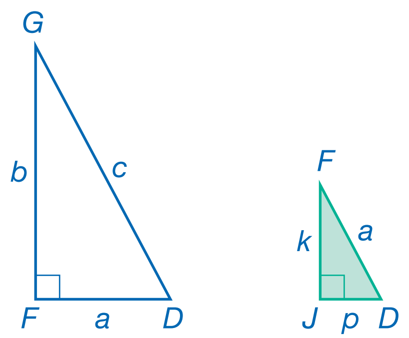
EXAMPLE B Right triangle DFG is shown on the left. Altitude $\overline{FJ}$ divides it into two right triangles, as shown on the right. Use these figures to derive the Pythagorean Theorem.



1 Compare $\triangle DFG$ and $\triangle DJF$.

Triangle DFG contains $\angle D$ and a right angle. Triangle DJF also contains $\angle D$ and a right angle. So, $\triangle DFG \sim \triangle DJF$ according to the Angle-Angle Similarity Postulate.

Copy $\triangle DJF$ and show the triangles with the same orientation to make comparing them easier.



Corresponding sides of similar triangles are proportional, so ratios of corresponding sides can be set equal to each other. Substitute the variables for each side length.

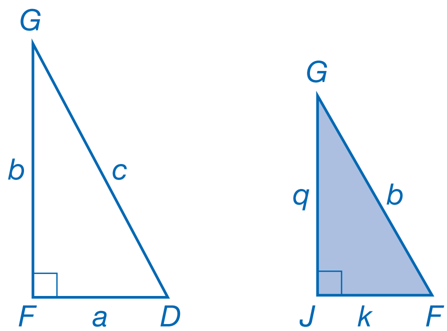
$\frac{GD}{FD} = \frac{FD}{JD}$ Substitute a, c, and p.

$\frac{c}{a} = \frac{a}{p}$ Cross multiply.

$cp = a^2$

2 Compare $\triangle DFG$ and $\triangle FJG$.

Triangles DFG and FJG both contain $\angle G$ and a right angle. So, $\triangle DFG \sim \triangle FJG$ according to AA~.



$\frac{GD}{GF} = \frac{GF}{GJ}$

$\frac{c}{b} = \frac{b}{q}$

$cq = b^2$

3 Derive the Pythagorean Theorem.

Examine $\overline{DG}$.

$DG = DJ + JG$

$c = p + q$

$a^2 = cp$ and $b^2 = cq$, so:

$a^2 + b^2 = cp + cq$ Factor out c.

$a^2 + b^2 = c(p + q)$ Substitute c for $p + q$.

$a^2 + b^2 = c(c)$

▶ $a^2 + b^2 = c^2$

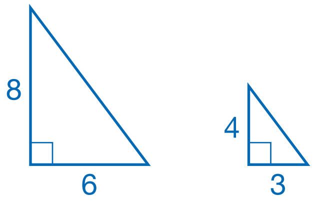
DISCUSS

Could you derive the Pythagorean Theorem if $\triangle DFG$ were not a right triangle?

Practice

Based on the side lengths and angle measures given or indicated, determine if the triangles can be proven similar. If so, state the postulate or theorem that applies.

1.



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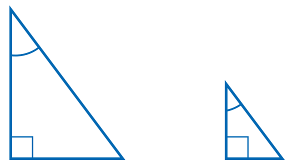
2.



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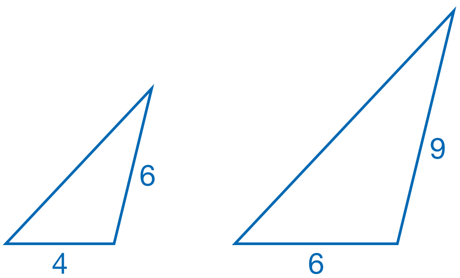
HINT Identical angles marks indicate congruent angles.

3.



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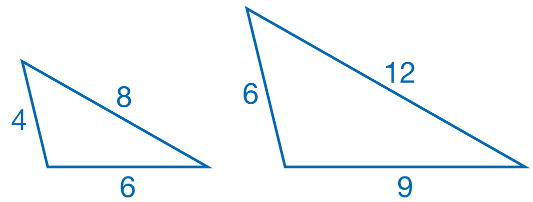
4.



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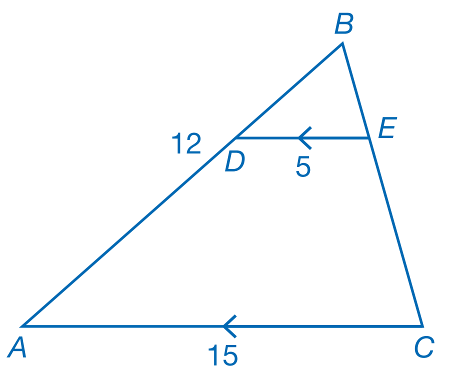
Choose the best answer.

5. In $\triangle ABC$, $\overline{DE}$ is parallel to $\overline{AC}$. If $BA = 12$, what is the value of BD?



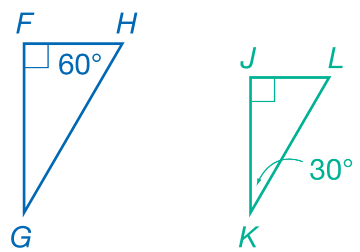
A. 2 units

B. 3 units

C. 4 units

D. 5 units

6. Using the information given in the figures below, which of the following allows you to prove that $\triangle FGH \sim \triangle JKL$?



A. AA~ Postulate

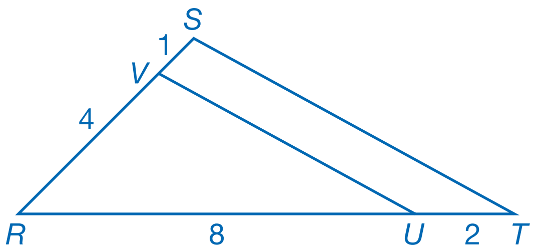
B. SSS~ Theorem

C. SAS~ Theorem

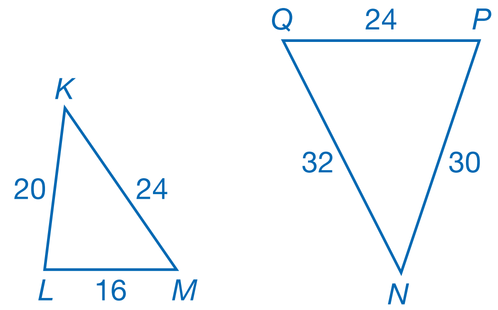
D. Pythagorean Theorem

For questions 7 and 8, determine if each pair of triangles is similar or not. Explain which theorem you used and which sides or angles you compared to determine your answer.

7. $\triangle RST$ and $\triangle RVU$



8. $\triangle KLM$ and $\triangle NPQ$



Solve.

9. COMPARE Ling went hiking one day. She measured the angle of the sun to be 55° from the horizon. At that time, her shadow was 3.5 feet long, and she is 5 feet tall. She measured the shadow of a tree to be 14.7 feet long. How tall is the tree?